Jacobi Elliptic Function Rational Expansion Method with Symbolic Computation to Construct New Doubly-periodic Solutions of Nonlinear Evolution Equations

Yong Chen[a,b,c], Qi Wang[c,d], and Biao Li[c,d]

[a] Department of Mathematics, Ningbo University, Ningbo 315211, China
[b] Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, China
[c] Department of Applied Mathematics, Dalian University of Technology, Dalian 116024, China
[d] MM Key Lab, Chinese Academy of Sciences, Beijing 100080, China

Reprint requests to Y. Ch.; E-mail: chenyong18@sjtu.edu.cn.

Z. Naturforsch. **59a,** 529 – 536 (2004); received May 3, 2004

A new Jacobi elliptic function rational expansion method is presented by means of a new general ansatz and is very powerful, with aid of symbolic computation, to uniformly construct more new exact doubly-periodic solutions in terms of rational form Jacobi elliptic function of nonlinear evolution equations (NLEEs). We choose a (2+1)-dimensional dispersive long wave equation to illustrate the method. As a result, we obtain the solutions found by most existing Jacobi elliptic function expansion methods and find other new and more general solutions at the same time. When the modulus of the Jacobi elliptic functions m $\to$ 1 or 0, the corresponding solitary wave solutions and trigonometric function (singly periodic) solutions are also found.

Key words: (2+1)-dimensional Dispersive Long Wave Equation; Jacobi Elliptic Functions; Travelling Wave Solution; Soliton Solution; Periodic Solution.

1. Introduction

Many activities aim at finding methods for exact solutions of nonlinear evolutions equations, such as Bäcklund transformation, Darboux transformation, variable separation approach, various tanh methods, Painleve method, generalized hyperbolic-function method, homogeneous balance method, similarity reduction method and so on [1 – 10].

For that there are some relations between the elliptic functions (e.g., Jacobi elliptic functions and Weierstrass elliptic functions) and some nonlinear evolution equations (NLEEs). Many authors have paid attention to this subject to know whether elliptic functions can be used to express the exact solutions of more nonlinear equations. Recently, Liu et al. [11] presented a Jacobi elliptic function expansion method that used three Jacobi elliptic functions to express exact solutions of some NLEEs. Fan [12] extended the Jacobi elliptic function method to some NLEEs and, in particular, special-type nonlinear equations for constructing their doubly periodic wave solutions. Such equations cannot be directly dealt with by this method and require some kinds of pre-possessing techniques. Yan [13] further developed an extended Jacobi elliptic function expansion method by using 12 Jacobi elliptic functions. Based on the above idea, a new Jacobi elliptic function rational expansion method is presented by means of a new general ansatz and is more powerful than the above Jacobi elliptic function method [12 – 13] to uniformly construct more new exact doubly-periodic solutions in terms of rational formal Jacobi elliptic functions of NLEEs. For illustration, we apply the proposed method to the (2+1)-dimensional dispersive long wave equation(DLWE), which reads

$$u_{yt} + v_{xx} + (uu_x)_y = 0, \qquad (1.1.1)$$

$$v_t + u_x + (uv)_x + u_{xxy} = 0. \qquad (1.1.2)$$

The (2+1)-dimensional DLWE (1) was first derived by Boiti et al. [14] as a compatibility for a "weak" Lax pair. Recently considerable effort has been devoted to the study of this system. In [15], Paquin and Winternitz showed that the symmetry algebra of the (2+1)-dimensional DLWE (1) is infinite-dimensional and possesses a Kac-Moody-Virasoro structure. Some special similarity solutions are also given in [15] by using symmetry algebra and classical theoretical analysis. The more general symmetry algebra, w_∞ symmetry algebra, is given in [16]. Lou [17] has given nine types of the two-dimensional partial differential equation reductions and thirteen types of the ordinary differential equation reductions by means of the direct and nonclassical method. The systems (1) have no Painleve property, though they are Lax or IST integrable [18]. More recently, Tang et al. [19], by means of the variable separation approach, the abundant localized coherent structures of the system (1) have given out. In [20], the possible chaotic and fractal localized structures are revealed for the system (1).

This paper is organized as follows: In Sect. 2 we summarize the Jacobi elliptic function rational expansion method. In Sect. 3 we apply the generalized method to (2+1)-dimensional dispersive long wave equations and bring out many solutions. Conclusions will be presented in Section 4.

2. Summary of the Jacobi Elliptic Function Rational Expansion Method

In the following we outline the main steps of our method:

Step 1. For a nonlinear evolution equation with physical fields $u_i(x,y,t)$ in three variables x, y, t,

$$F_i(u_i,u_{it},u_{ix},u_{iy},u_{itt},u_{ixt},u_{iyt},u_{ixx},u_{iyy},u_{ixy},\cdots)=0, \tag{2.1}$$

we use the transformation

$$u_i(x,y,t)=u_i(\xi), \quad \xi=k(x+ly-\lambda t), \tag{2.2}$$

where k, l and λ are constants to be determined later. Then (2.1) is reduced to the nonlinear ordinary differential equation (ODE)

$$G_i(u_i,u_i',u_i'',\cdots)=0. \tag{2.3}$$

Step 2. We introduce a new ansatz in terms of a finite Jacobi elliptic function rational expansion:

$$u_i(\xi)=a_{i0}+\sum_{j=1}^{m_i}\Big(a_{ij}\frac{\mathrm{sn}^j(\xi)}{(\mu\,\mathrm{sn}(\xi)+1)^j} + b_{ij}\frac{\mathrm{sn}^{j-1}(\xi)\mathrm{cn}(\xi)}{(\mu\,\mathrm{sn}(\xi)+1)^j}\Big). \tag{2.4}$$

Notice that

$$\frac{\mathrm{d}u_i}{\mathrm{d}\xi}=\sum_{j=1}^{m_i}\frac{\mathrm{dn}(\xi)\left(a_{ij}j(\mathrm{sn}(\xi))^{j-1}\mathrm{cn}(\xi)-b_{ij}(\mathrm{sn}(\xi))^{j-2}-b_{ij}\mu(\mathrm{sn}(\xi))^{j-1}\right)}{(\mu\,\mathrm{sn}(\xi)+1)^{j+1}} + \sum_{j=1}^{m_i}\frac{\mathrm{dn}(\xi)\left(b_{ij}j(\mathrm{sn}(\xi))^{j-2}-b_{ij}j(\mathrm{sn}(\xi))^{j}\right)}{(\mu\,\mathrm{sn}(\xi)+1)^{j+1}}, \tag{2.5}$$

where $\mathrm{sn}\xi$, $\mathrm{cn}\xi$, $\mathrm{dn}\xi$, $\mathrm{ns}\xi$, $\mathrm{cs}\xi$, $\mathrm{ds}\xi$ etc. are Jacobi elliptic functions, which are double periodic and posses the following properties:

1. Properties of triangular functions:

$$\mathrm{cn}^2\xi+\mathrm{sn}^2\xi=\mathrm{dn}^2\xi+\mathrm{m}^2\mathrm{sn}^2\xi=1, \tag{2.6}$$

2. Derivatives of the Jacobi elliptic functions:

$$\begin{aligned}\mathrm{sn}'\xi&=\mathrm{cn}\xi\,\mathrm{dn}\xi,\\ \mathrm{cn}'\xi&=-\mathrm{sn}\xi\,\mathrm{dn}\xi,\\ \mathrm{dn}'\xi&=-m^2\mathrm{sn}\xi\,\mathrm{cn}\xi,\end{aligned} \tag{2.7}$$

where m is a modulus. The Jacobi-Glaisher functions for elliptic functions can be found in [21–23].

Step 3. The underlying mechanism for a series of fundamental solutions, such as polynomial, exponential, solitary wave, rational, triangular periodic, Jacobi and Weierstrass doubly periodic solutions to occur, is that different effects that act to change the wave forms in many nonlinear equations, i. e. dispersion, dissipation and nonlinearity, either separately or in various combinations are able to balance out. We define the degree of $u_i(\xi)$ as $D[u_i(\xi)]=n_i$, which gives rise to the degrees of other expressions as

$$\begin{aligned}&D[u_i^{(\alpha)}]=n_i+\alpha,\\ &D[u_i^{\beta}(u_j^{(\alpha)})^s]=n_i\beta+(\alpha+n_j)s.\end{aligned} \tag{2.8}$$

Therefore we can get the value of m_i in (2.4). If n_i is a nonnegative integer, then we first make the transformation $u_i=\omega^{n_i}$.

Step 4. Substitute (2.4) into (2.3) along with (2.5) and (2.7), and then set all coefficients of $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$, $(i = 1,2,\cdots;\ j = 0,1)$ to zero, to get an over-determined system of algebraic equations with respect to λ, l, μ, a_{i0}, a_{ij} and b_{ij} $(i = 1,2,\cdots;\ j = 1,2,\cdots,m_i)$.

Step 5. Solving the over-determined system of nonlinear algebraic equations by use of Maple, we would end up with explicit expressions for λ, l, μ, k, a_{i0}, a_{ij} and b_{ij}, $(i = 1,2,\cdots;\ j = 1,2,\cdots,m_i)$.

In this way, we can get double periodic solutions with Jacobi elliptic functions. Since:

$$\lim_{m\to 1}\mathrm{sn}\xi = \tanh\xi,\quad \lim_{m\to 1}\mathrm{cn}\xi = \mathrm{sech}\xi,\quad \lim_{m\to 1}\mathrm{dn}\xi = \mathrm{sech}\xi \tag{2.9.1}$$

$$\lim_{m\to 0}\mathrm{sn}\xi = \sin\xi,\quad \lim_{m\to 0}\mathrm{cn}\xi = \cos\xi,\quad \lim_{m\to 0}\mathrm{dn}\xi = 1, \tag{2.9.2}$$

u_1 and u_2 degenerate respectively as the following form:

1. solitary wave solutions:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\Big(a_{ij}\frac{\tanh^j(\xi)}{(\mu\tanh(\xi)+1)^j} + b_{ij}\frac{\tanh^{j-1}(\xi)\mathrm{sech}(\xi)}{(\mu\tanh(\xi)+1)^j}\Big). \tag{2.10}$$

2. triangular function formal solution:

$$u_i(\xi) = a_{i0} + \sum_{j=1}^{m_i}\Big(a_{ij}\frac{\sin^j(\xi)}{(\mu\sin(\xi)+1)^j} + b_{ij}\frac{\sin^{j-1}(\xi)\cos(\xi)}{(\mu\sin(\xi)+1)^j}\Big). \tag{2.11}$$

So the Jacobi elliptic function rational expansion method is more powerful than the method by Liu et al. [11], the method by Fan [12], and the method extended by Yan [13]. The solutions which contain solitary wave solutions, singular solitary solutions and triangular function formal solutions can be obtained by the extended method.

Remark1: If we replace the Jacobi elliptic functions $\mathrm{sn}(\xi)$ and $\mathrm{cn}(\xi)$ in the ansatz (2.4) with other pairs of Jacobi elliptic functions, such as $\mathrm{sn}(\xi)$ and $\mathrm{dn}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{cs}(\xi)$; $\mathrm{ns}(\xi)$ and $\mathrm{ds}(\xi)$; $\mathrm{sc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{dc}(\xi)$ and $\mathrm{nc}(\xi)$; $\mathrm{sd}(\xi)$ and $\mathrm{nd}(\xi)$; $\mathrm{cd}(\xi)$ and $\mathrm{nd}(\xi)$ [13–14], it is necessary to point out that the above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial to close the computation. Therefore other new double periodic wave solutions, solitary wave solutions, and triangular functional solutions can be obtained for some systems. For simplicity we omit them here.

3. Exact Solutions of the (2+1)-dimensional Dispersive Long Wave Equation (DLWE)

Let us consider the (2+1)-dimensional dispersive long wave equation (DLWE), i. e.

$$\begin{cases} u_{yt} + v_{xx} + (uu_x)_y = 0, \\ v_t + u_x + (uv)_x + u_{xxy} = 0. \end{cases} \tag{3.1}$$

According to the above method, to seek travelling wave solutions of (3.1) we make the transformation

$$\begin{aligned} &u(x,y,t) = \phi(\xi),\\ &v(x,y,t) = \sigma(\xi),\\ &\xi = x + ly - \lambda t, \end{aligned} \tag{3.2}$$

where l and λ are constants to be determined later, and thus (3.1) become

$$\begin{cases} -\lambda l\phi'' + \sigma'' + l\phi'^2 + l\phi\phi'' = 0, \\ -\lambda\sigma' + \phi' + (\phi\sigma)' + l\phi''' = 0. \end{cases} \tag{3.3}$$

According to Step 1 in Section 2, by balancing $\phi'''(\xi)$ and $(\sigma(\xi)\phi(\xi))'$ in (3.3), and by balancing $\sigma''(\xi)$ and $\phi(\xi)\phi''(\xi)$ in (3.3), we suppose that (3.3) has the following formal solutions

$$\begin{cases} \phi(\xi) = a_0 + a_1\dfrac{\mathrm{sn}(\xi)}{\mu\mathrm{sn}(\xi)+1} + b_1\mathrm{cn}(\xi)\mu\mathrm{sn}(\xi)+1, \\ \sigma(\xi) = A_0 + A_1\dfrac{\mathrm{sn}(\xi)}{\mu\mathrm{sn}(\xi)+1} + B_1\dfrac{\mathrm{cn}(\xi)}{\mu\mathrm{sn}(\xi)+1} \\ \qquad + A_2\dfrac{\mathrm{sn}^2(\xi)}{(\mu\mathrm{sn}(\xi)+1)^2} + B_2\dfrac{\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{(\mu\mathrm{sn}(\xi)+1)^2}, \end{cases} \tag{3.4}$$

where a_0, a_1, b_1, A_0, A_1, A_2, B_1, B_2 are constants to be determined later.

With the aid of *Maple*, substituting (3.4) along with (2.6) and (2.7) into (3.3), yields a set of algebraic equations for $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$, $(i = 0,1,\cdots;\ j = 0,1)$. Setting the coefficients of these terms $\mathrm{sn}^i(\xi)\mathrm{cn}^j(\xi)$

to zero yields a set of over-determined algebraic equations with respect to $a_0, a_1, b_1, A_0, A_1, B_1, A_2, B_2$, l and λ.

By use of the *Maple* soft package "Charsets" by Dongming Wang, which is based on the Wu-elimination method [24], solving the over-determined algebraic equations, we get the following results.

Case 1.

$$A_1 = B_1 = \mu = 0,\ a_0 = \lambda,\ a_1 = \pm m,\ b_1 = \pm im,\\ A_0 = l-1,\ A_2 = -im^2,\ B_2 = \pm ilm^2. \quad (3.5)$$

Case 2.

$$b_1 = A_1 = B_1 = B_2 = \mu = 0,\ a_0 = \lambda,\\ a_1 = \pm 2m,\ A_0 = l-1+lm^2,\ A_2 = -2im^2. \quad (3.6)$$

Case 3.

$$a_1 = A_1 = B_1 = B_2 = \mu = 0, \quad a_0 = \lambda, \quad b_1 = \pm 2im, \quad A_0 = l-1, \quad A_2 = -2im^2. \quad (3.7)$$

Case 4.

$$\begin{aligned} &a_0 = \frac{8\lambda m^2 + 2\lambda \pm i\sqrt{20m^2+5}}{2(1+4m^2)}, \quad a_1 = \pm\frac{\sqrt{20m^2+5}}{4}, \quad b_1 = \pm\frac{\sqrt{-1-4m^2}}{2},\\ &A_0 = \frac{24lm^2 - 16m^2 + l - 4}{4(1+4m^2)}, \quad A_1 = \frac{i(2m^2+3)l}{4}, \quad A_2 = -\frac{5}{4}lm^2 - \frac{5}{16}l,\\ &B_1 = \frac{\pm il\sqrt{-1-4m^2}\sqrt{20m^2+5}}{4(1+4m^2)}, \quad B_2 = \pm\frac{l\sqrt{-1-4m^2}\sqrt{20m^2+5}}{8}, \quad \mu = \pm\frac{1}{2}i. \end{aligned} \quad (3.8)$$

Case 5.

$$\begin{aligned} &a_0 = \frac{8m^6\lambda + 20m^4\lambda + 10\lambda m^2 - 2\lambda \pm \sqrt{24m^6+48m^4-3+12m^2}\sqrt{-(m^2+1)(2m^2-1)}}{2(4m^6+10m^4+5m^2-1)},\\ &a_1 = \pm\frac{\sqrt{24m^6+48m^4-3+12m^2}}{4(m^2+1)}, \quad b_1 = \pm\frac{\sqrt{-(m^2+1)(4m^4-1+6m^2)}}{2+2m^2},\\ &A_0 = \frac{-16m^6+32lm^6-40m^4+44m^4l+4lm^2-20m^2+4+l}{4(4m^6+10m^4+5m^2-1)}, \quad \mu = \pm\frac{\sqrt{-(m^2+1)(2m^2-1)}}{2+2m^2},\\ &A_1 = \pm\frac{(2m^4+6m^2+1)l\sqrt{-(m^2+1)(2m^2-1)}}{4(m^4+2m^2+1)}, \quad A_2 = -3\frac{l(8m^6+16m^4-1+4m^2)}{16(m^4+2m^2+1)},\\ &B_1 = \pm\frac{l\sqrt{-(m^2+1)(4m^4-1+6m^2)}\sqrt{24m^6+48m^4-3+12m^2}\sqrt{-(m^2+1)(2m^2-1)}}{4(4m^8+14m^6+15m^4+4m^2-1)},\\ &B_2 = \pm\frac{l\sqrt{-(m^2+1)(4m^4-1+6m^2)}\sqrt{24m^6+48m^4-3+12m^2}}{8(m^4+2m^2+1)}. \end{aligned} \quad (3.9)$$

Case 6.

$$\begin{aligned} &a_0 = \frac{\pm(\mu^3-\mu) + \sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\lambda}{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}}, \quad a_1 = \pm\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}, \quad b_1 = \pm\sqrt{-m^2+\mu^2},\\ &A_0 = -\frac{-l\mu^4+2l\mu^2m^2-\mu^2+m^2-lm^2}{-\mu^2+m^2}, \quad A_1 = -lm^2\mu + 2l\mu^3 - l\mu, \quad A_2 = -lm^2 - l\mu^4 + l\mu^2 + l\mu^2m^2,\\ &B_1 = \pm\frac{l\sqrt{-m^2+\mu^2}\mu(-1+\mu^2)}{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}}, \quad B_2 = \pm l\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\sqrt{-m^2+\mu^2}. \end{aligned} \quad (3.10)$$

Case 7.

$$
\begin{aligned}
a_0 &= \frac{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\mu+\mu m^2\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}-2\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\mu^3}{-m^2-\mu^4+\mu^2+\mu^2m^2}, \\
&\frac{-\lambda m^2-\mu^4\lambda+\mu^2\lambda+\mu^2m^2\lambda}{-m^2-\mu^4+\mu^2+\mu^2m^2}, \quad a_1=-2\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}, \\
b_1 &= B_1=B_2=0, A_0=-\frac{\mu^2-m^2-\mu^4+\mu^2m^2-6l\mu^2m^2+3l\mu^4+m^4l-2l\mu^6+3lm^2\mu^4+lm^2}{-m^2-\mu^4+\mu^2+\mu^2m^2}, \\
A_1 &= -2l\mu-2lm^2\mu+4l\mu^3, \quad A_2=-2lm^2+2l\mu^2m^2-2l\mu^4+2l\mu^2.
\end{aligned} \tag{3.11}
$$

From (3.2), (3.4) and Case 1 – 7, we obtain the following solutions for (3.1).

Family 1. From (3.5) we obtain the following rational formal doubly periodic solutions for the DLWE:

$$u_1(x,y,t)=\lambda\pm m\,\mathrm{sn}(\xi)\pm im\,\mathrm{cn}(\xi), \quad v_1(x,y,t)=l-1-lm^2\mathrm{sn}^2(\xi)\pm ilm^2\mathrm{sn}(\xi)\mathrm{cn}(\xi), \tag{3.12}$$

where $\xi=x+ly-\lambda t$, l and λ are arbitrary constants.

Family 2. From (3.6) we obtain the following rational formal doubly periodic solutions for the DLWE:

$$u_2(x,y,t)=\lambda\pm 2m\,\mathrm{sn}(\xi), \quad v_2(x,y,t)=l-1+lm^2-2lm^2\mathrm{sn}^2(\xi), \tag{3.13}$$

where $\xi=x+ly-\lambda t$, l and λ are arbitrary constants.

Family 3. From (3.7), we obtain the following rational formal doubly periodic solutions for the DLWE:

$$u_3(x,y,t)=\lambda\pm 2im\,\mathrm{cn}(\xi), \quad v_3(x,y,t)=l-1-2lm^2\mathrm{sn}^2(\xi), \tag{3.14}$$

where $\xi=x+ly-\lambda t$, l and λ are arbitrary constants.

Family 4. From (3.8), we obtain the following rational formal doubly periodic solutions for the DLWE:

$$
\begin{aligned}
u_4(x,y,t) &= \frac{8\lambda m^2+2\lambda\pm i\sqrt{20m^2+5}}{2(1+4m^2)}\pm\frac{\sqrt{20m^2+5}\,\mathrm{sn}(\xi)}{\pm 2i\,\mathrm{sn}(\xi)+4}\pm\frac{\sqrt{-1-4m^2}\,\mathrm{cn}(\xi)}{\pm i\,\mathrm{sn}(\xi)+2}, \\
v_4(x,y,t) &= \frac{24lm^2-16m^2+l-4}{4(1+4m^2)}+\frac{i(2m^2+3)l\,\mathrm{sn}(\xi)}{\pm 2i\,\mathrm{sn}(\xi)+4}\pm\frac{il\,\mathrm{cn}(\xi)}{\pm 2i\,\mathrm{sn}(\xi)+4} \\
&\quad +\frac{\left(-\frac{5}{4}lm^2-\frac{5}{16}l\right)\mathrm{sn}^2(\xi)}{\left(\pm\frac{1}{2}i\,\mathrm{sn}(\xi)+1\right)^2}\pm\frac{l\sqrt{-1-4m^2}\sqrt{20m^2+5}\,\mathrm{sn}(\xi)\mathrm{cn}(\xi)}{8\left(\pm\frac{1}{2}i\,\mathrm{sn}(\xi)+1\right)^2},
\end{aligned} \tag{3.15}
$$

where $\xi=x+ly-\lambda t$, l and λ are arbitrary constants.

Family 5. From (3.9), we obtain the following rational formal doubly periodic solutions for the DLWE:

$$u_5(x,y,t)=a_0\pm\frac{\sqrt{24m^6+48m^4-3+12m^2}\,\mathrm{sn}(\xi)}{(m^2+1)\left(\pm 2\frac{\sqrt{-(m^2+1)(2m^2-1)}\mathrm{sn}(\xi)}{1+m^2}+4\right)}\pm\frac{\sqrt{-(m^2+1)(4m^4-1+6m^2)}\,\mathrm{cn}(\xi)}{(2+2m^2)\left(\pm\frac{\sqrt{-(m^2+1)(2m^2-1)}\mathrm{sn}(\xi)}{2+2m^2}+1\right)},$$

$$\begin{aligned} v_5(x,y,t) = A_0 &\pm \frac{(2m^4+6m^2+1)l\sqrt{-(m^2+1)(2m^2-1)}\operatorname{sn}(\xi)}{(m^4+2m^2+1)\left(\pm 2\frac{\sqrt{-(m^2+1)(2m^2-1)}\operatorname{sn}(\xi)}{1+m^2}+4\right)} \\ &- \frac{3l(8m^6+16m^4-1+4m^2)\operatorname{sn}^2(\xi)}{16(m^4+2m^2+1)\left(\pm\frac{\sqrt{-(m^2+1)(2m^2-1)}\operatorname{sn}(\xi)}{2+2m^2}+1\right)^2} \\ &\pm \frac{l\sqrt{-(m^2+1)(4m^4-1+6m^2)}\sqrt{24m^6+48m^4-3+12m^2}\sqrt{-(m^2+1)(2m^2-1)}\operatorname{cn}(\xi)}{(4m^8+14m^6+15m^4+4m^2-1)\left(\pm 2\frac{\sqrt{-(m^2+1)(2m^2-1)}\operatorname{sn}(\xi)}{1+m^2}+4\right)} \\ &\pm \frac{l\sqrt{-(m^2+1)(4m^4-1+6m^2)}\sqrt{24m^6+48m^4-3+12m^2}\operatorname{sn}(\xi)\operatorname{cn}(\xi)}{8(m^4+2m^2+1)\left(\pm\frac{\sqrt{-(m^2+1)(2m^2-1)}\operatorname{sn}(\xi)}{2+2m^2}+1\right)^2}, \end{aligned} \tag{3.16}$$

where $\xi = x+ly-\lambda t$, $a_0 = \frac{8m^6\lambda+20m^4\lambda+10\lambda m^2-2\lambda\pm\sqrt{24m^6+48m^4-3+12m^2}\sqrt{-(m^2+1)(2m^2-1)}}{2(4m^6+10m^4+5m^2-1)}$, $A_0 = \frac{-16m^6+32lm^6-40m^4+44m^4l+4lm^2-20m^2+4+l}{4(4m^6+10m^4+5m^2-1)}$, l and λ are arbitrary constants.

Family 6. From (3.10) we obtain the following rational formal doubly periodic solutions for the DLWE:

$$u_6(x,y,t) = \pm\frac{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\operatorname{sn}(\xi)}{\mu\operatorname{sn}(\xi)+1} \pm \frac{\sqrt{-m^2+\mu^2}\operatorname{cn}(\xi)}{\mu\operatorname{sn}(\xi)+1} + \frac{\pm(\mu^3-\mu)+\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\lambda}{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}},$$

$$\begin{aligned} v_6(x,y,t) = &-\frac{-l\mu^4+2l\mu^2m^2-\mu^2+m^2-lm^2}{-\mu^2+m^2} + \frac{(-lm^2\mu+2l\mu^3-l\mu)\operatorname{sn}(\xi)}{\mu\operatorname{sn}(\xi)+1} \\ &\pm \frac{l\sqrt{-m^2+\mu^2}\mu(-1+\mu^2)\operatorname{cn}(\xi)}{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}(\mu\operatorname{sn}(\xi)+1)} + \frac{(-lm^2-l\mu^4+l\mu^2+l\mu^2m^2)\operatorname{sn}^2(\xi)}{(\mu\operatorname{sn}(\xi)+1)^2} \\ &\pm \frac{l\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\sqrt{-m^2+\mu^2}\operatorname{sn}(\xi)\operatorname{cn}(\xi)}{(\mu\operatorname{sn}(\xi)+1)^2}, \end{aligned} \tag{3.17}$$

where $\xi = x+ly-\lambda t$, l and λ are arbitrary constants.

Family 7. From (3.11), we obtain the following rational formal doubly periodic solutions for the DLWE:

$$\begin{aligned} u_7(x,y,t) &= a_0 + 2\frac{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\operatorname{sn}(\xi)}{\mu\operatorname{sn}(\xi)+1}, \\ v_7(x,y,t) &= A_0 + \frac{(-2l\mu-2lm^2\mu+4l\mu^3)\operatorname{sn}(\xi)}{\mu\operatorname{sn}(\xi)+1} + \frac{(-2lm^2+2l\mu^2m^2-2l\mu^4+2l\mu^2)\operatorname{sn}^2(\xi)}{(\mu\operatorname{sn}(\xi)+1)^2}, \end{aligned} \tag{3.18}$$

where $\xi = x+ly-\lambda t$, $a_0 = \frac{\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\mu+\mu m^2\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}-2\sqrt{m^2+\mu^4-\mu^2-\mu^2m^2}\mu^3}{-m^2-\mu^4+\mu^2+\mu^2m^2}$, $A_0 = -\frac{\mu^2-m^2-\mu^4+\mu^2m^2-6l\mu^2m^2+3l\mu^4+m^4l-2l\mu^6+3lm^2\mu^4+lm^2}{-m^2-\mu^4+\mu^2+\mu^2m^2}$, l and λ are arbitrary constants.

Remark 2:

1) The solutions (3.12) reproduce the solution (15) in [22], when $A_0 = \frac{2(C_1+1)(m^2+1)}{2+2m^2-\lambda^2} - 1$.

2) The other solutions obtained here, to our knowledge, are all new families of doubly periodic solutions of the DLWE.

3) Three figures are drawn to illustrate the properties for some solutions.

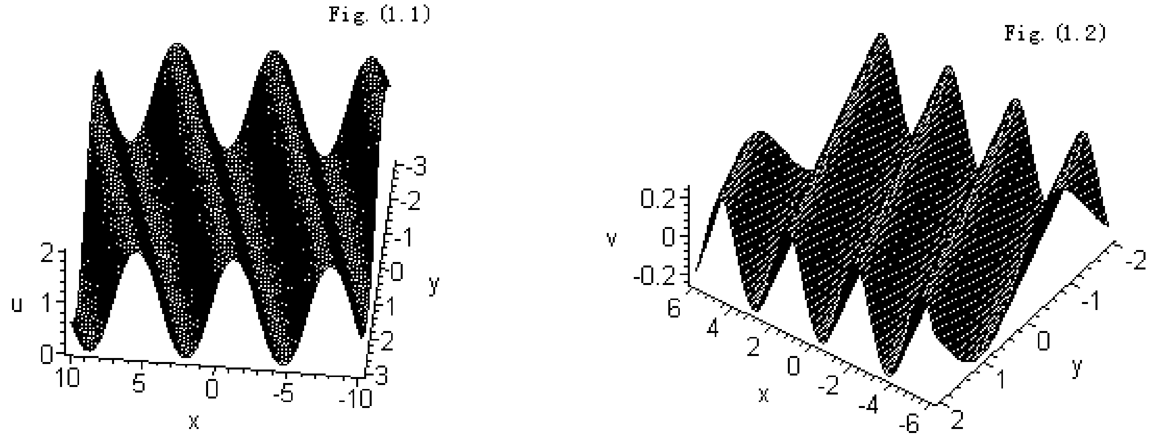

Fig. 1. The doubly periodic solutions u_2 and v_2, where $l = 1$, $\lambda = 1$, $m = \frac{1}{2}$, $t = 0$.

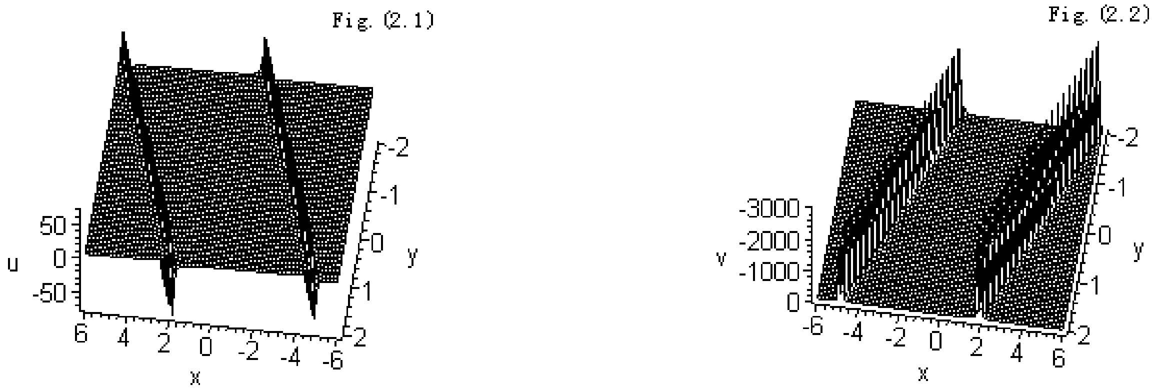

Fig. 2. The doubly periodic solutions u_6 and v_6, where $l = 1$, $\lambda = 1$, $m = \frac{1}{2}$, $t = 0$, $\mu = 2$.

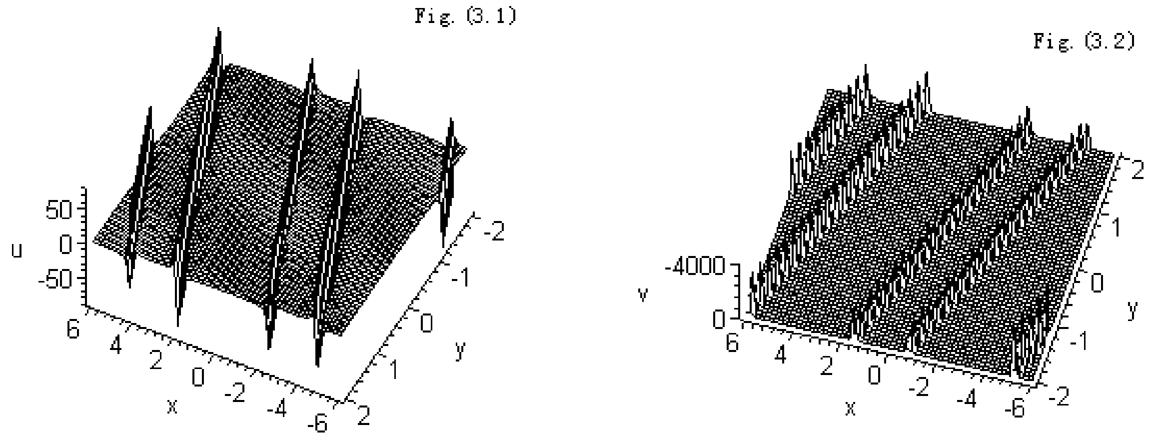

Fig. 3. The doubly periodic solutions u_7 and v_7, where $l = 1$, $\lambda = 1$, $m = \frac{1}{2}$, $t = 0$, $\mu = 2$.

4. Summary and Conclusions

In this paper we have presented the new Jacobi elliptic function rational expansion method. The method is more powerful than the method proposed recently by Liu [11], Fan [12] and Yan [13]. The (2+1)-dimensional dispersive long wave equation (DLWE) is chosen to illustrate the method such that seven families of new Jacobi elliptic function solutions are obtained. When the modulus $m \to 1$, some of these obtained solutions degenerate as solitary wave solutions. The algorithm can also be applied to many nonlinear evolution equations in mathematical physics. Further work about various extensions and improvement of the Jacobi function method is needed to find more general ansätze or the more general subequation.

Acknowledgements

The work is supported by the Outstanding Youth Foundation (No.19925522).

[1] M. J. Ablowitz and P. A. Clarkson, Soliton, Nonlinear Evolution Equations and Inverse Scattering, Cambridge University Press, New York 1991.
[2] V. B. Matveev and M. A. Salle, Darbooux Transformation and Soliton, Spinger-Verlag, Berlin 1991.
[3] S. Y. Lou and J. Z. Lu, J. Phys. A **29**, 4029 (1996); S. Y. Lou, Phys. Lett. A **277**, 94 (2000); S. Y. Lou and H. Y. Ruan, J. Phys. A **34**, 305 (2001).
[4] R. Hirta and J. Satsuma, Phys. Lett. A **85**, 407 (1981).
[5] H. W. Tam, W. X. Ma, X. B. Hu, and D. L. Wang, J. Phys. Soc. Japan **69**, 45 (2000).
[6] Y. Chen, B. Li and H. Q. Zhang, Chaos, Solitons and Fractals **17**, 693 (2003); Y. Chen, Z. Y. Yan and H. Q. Zhang, Theor. Math. Phys. **132**, 970 (2002); Y. Chen and Y. Zheng, IJMPC, **14**, 601 (2003); B. Li, Y. Chen, and H. Q. Zhang, Phys. Lett. A **305**, 377 (2002).
[7] M. L. Wang, Phys Lett. A **216**, 67 (1996).
[8] E. J. Parkes and B. R. Duffy, Comp. Phys. Commun. **98**, 288 (1996); E. J. Parkes and B. R. Duffy, Phys. Lett. A **229**, 217 (1997).
[9] Y. T. Gao and B. Tian, Comput. Phys. Commun. **133**, 158 (2001).
[10] E. Fan Computer Physics Communications **153**, 17 (2003); E. Fan Chaos, Solitons and Fractals **15**, 559 (2003); E. Fan Physics Letters A **300**, 243 (2002).
[11] S. K. Liu, et al., Phys. Lett. A **290**, 72 (2001); S. K. Liu, et al., Phys. Lett. A **289**, 69 (2001).
[12] E. Fan, Engui, Hon, and Y. C. Benny, Phys. Lett. A **292**, 335 (2002); E. Fan, Engui, H. Q. Zhang, and Jian, Phys. Lett. A **305**, 383 (2002).
[13] Z. Y. Yan, Chaos, Solitons and Fractals **18**, 299 (2003); Z. Y. Yan, Chaos, Solitons and Fractals **15**, 575 (2003); Z. Y. Yan, Computer Physics Communications **148**, 30 (2002).
[14] P. Boiti, J. J. P. Leon, M. Manna, and F. Pempinelli, Inverse Problem **2**, 271 (1986); ibid. **3**, 371 (1987).
[15] G. Paquin and P. Winternitz, Physica D **46**, 122 (1990).
[16] S.-Y. Lou, J. Phys. A: Math. Gen. **27**, 3235 (1994).
[17] S.-Y. Lou, Math. Methods Appl. Sci. **18**, 789 (1995).
[18] S.-Y. Lou, Phys. Lett. A **176**, 9 (1993).
[19] X.-Y. Tang, C.-L. Chen, and S.-Y. Lou, J. Phys. A: Math. Gen. **35**, L293 (2002).
[20] X.-Y. Tang and S.-Y. Lou, Chaos, Solitons and Fractals **14**, 1451 (2002).
[21] K. Chandrasekharan, Elliptic Function, Springer-Verlag, Berlin 1978.
[22] D. V. Patrick, Elliptic Function and Elliptic Curves, At the University Press, Cambridge 1973.
[23] Z. X. Wang and X. J. Xia, Special Functions, World Scientific, Singapore 1989.
[24] W. Wu, Algorithms and Computation, Springer-Verlag, Berlin 1994, p.1.